Homework 2

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1. Because $Z_{12}$ is cyclic and its generators denoted by $a$ could be $1,5,7,11$, $\psi$ is completely determined by its value $g$ on $\psi(a)$, that is $\psi(a) = g$. In order to be homomorphic, $|g|$ must divide $|a| = 12$ otherwise after a cycle $\psi(a) = g \neq g^{12} = \psi(a^{12})$. The $g$ in $Z_{12}$ whose orders divide 12 are $1,2,3,4,6,12$. Thus the endomorphisms are

$$\psi(x) = y \text{ where } x \in \{1, 5, 7, 11\} \text{ and } y \in \{1, 2, 3, 4, 6, 0\}$$

The isomorphisms are $\psi(x) = 1$ where $x \in \{1, 5, 7, 11\}$ because only 1 generates $Z_{12}$

2. The additive group $Z$ of all integers is isomorphic to the subgroup of even integers under the isomorphism $f(x) = 2 \ast x$.

3. (a) $Z^+$ is an infinite cyclic group. The automorphism group is $\{\psi(1) = 1, \psi(1) = -1\}$.

(b) $G_{10}$ is a finite cyclic group. The automorphism group is $\{\psi(g_1) = g_2 \mid g_1, g_2 \text{ are generators of } G_{10}\}$.

(c) let $g,k$ be two distinct elements in the group $\{(13), (23),(12)\}$ and let $x,y$ be another two distinct ones in the same group. Then the automorphism group is $\{\psi(g) = x, \psi(k) = y\}$.

4. prove: because $Z^+$ is an infinite cyclic group, homomorphism from it is totally determined by $\psi(1) = g$. The homomorphism is isomorphic and surjective if $g = 1$ or -1. It’s injective otherwise.

5. prove: let $C$ denote the center of the group $G$. Let $g,c$ be some elements in $G,C$ respectively. By definition of center, $cg = gc \Rightarrow g^{-1}cg = c \in C$. Thus $C$ is normal.

6. typo

7. prove: let $\psi$ denote the map $\psi: G \rightarrow G$ defined by $g \rightarrow g^{-1}$. 

$\Rightarrow b^{-1}a^{-1} = \psi(ab) = \psi(a)\psi(b) = a^{-1}b^{-1}$. Thus $\psi$ is abelian

$\Leftarrow \psi(ab) = b^{-1}a^{-1} = a^{-1}b^{-1} = \psi(a)\psi(b)$. Thus it’s homomorphic.
8. prove:

(a) for \( n = 1 \) and \( m = 0 \), \( S_1 \) is a singleton set and \( S_0 \) is the empty set. They’re not isomorphic to each other.
(b) for \( n, m \geq 2 \), \( S_n \) and \( S_m \) have order \( n! \) and \( m! \) respectively. If \( n \neq m \), then \( n! \neq m! \). Thus there is no bijective map between \( S_n \) and \( S_m \). Hence they’re not isomorphic. Thus for \( n \neq m \), \( S_n \) and \( S_m \) are not isomorphic.

9. (a) No. e.g. \( 1 \sim 2, 2 \sim 3 \), but \( 1 \sim 3 \) is false.
(b) No. e.g. \( 1 \sim 2 \), but \( 2 \sim 1 \) is false.

10. prove:

(a) transitive: let \( a, b, c \in G \), \( a \sim b \), \( b \sim c \). \( a, b, c \) are distinct.
    \[
    \begin{align*}
    a \sim b & \Rightarrow b^{-1}a \in H \\
    b \sim c & \Rightarrow c^{-1}b \in H \\
    \text{Thus } c^{-1}bb^{-1}a & = c^{-1}a \in H \Rightarrow a \sim c
    \end{align*}
    \]
(b) symmetric: let \( a, b \in G \), \( a \sim b \). \( a, b \) are distinct.
    \[
    \begin{align*}
    a \sim b & \Rightarrow b^{-1}a \in H \Rightarrow a^{-1}b \in H \text{ because } H \text{ is a group } \Rightarrow b \sim a.
    \end{align*}
    \]
(c) reflexive: let \( a \in G \)
    \[
    a^{-1}a = e \in H
    \]
    Thus \( a \sim a \)

11. (a) prove: let \( a, b, c \in G \), \( a \sim b \), \( b \sim c \). \( a, b, c \) are distinct.

    i. transitive: \( a \sim b \Rightarrow \exists g \in G \text{ s.t. } gag^{-1} = b \)
        \[
        \begin{align*}
        b \sim c & \Rightarrow \exists k \in G \text{ s.t. } kbk^{-1} = c \\
        \text{Thus } kgag^{-1}k^{-1} & = c \Rightarrow a \sim c
        \end{align*}
        \]
    ii. symmetric: let \( a, b \in G \), \( a \sim b \). \( a, b \) are distinct.
        \[
        a \sim b \Rightarrow \exists g \in G \text{ s.t. } gag^{-1} = b \Rightarrow a = g^{-1}bg \Rightarrow b \sim a.
        \]
    iii. reflexive: let \( a \in G \)
        \[
        e^{-1}ae = a
        \]
    Thus \( a \sim a \)

(b) Elements in the center of \( G \).

12. infinite. Because \( | Z | = \infty \) and \( | nZ | = n \), \( | Z : nZ | = | Z | / | nZ | = \infty \)

13. prove: because both 3 and 5 are prime numbers, \( H \) and \( K \) are generated by some elements \( h, k \) that are not identities and they have the forms \( \{1, h, h^2\} \) and \( \{1, k, ..., k^4\} \) respectively, according to Corollary 6.13. \( h \) and \( k \) can not be identical because they have different orders. Thus the only element in the intersection of \( H \) and \( K \) is 1. Thus \( H \cap K = \{1\} \).
14. (a) prove: because index of subgroup $H$ is 2, there are only two right cosets, namely $H$ itself and $Hx$ where $x$ doesn’t belong to $H$. Thus $Hx=G-H$ by corollary 6.3.

Similarly, there’re two left cosets $H$ and $yH$ and $yH=G-H=Hx$.
Thus $H$ is normal by prop 6.18.

(b) $D_3$